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Chebyshev Approximation of Regulated Functions by Unisolvent Families

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1. INTRODUCTION

In this paper we consider the Chebyshev approximation of a large class of bounded functions, by a unisolvent family of functions. Let f be a unisolvent family (of degree n) of continuous functions on $[a, b]$. That is, if $x_1, \ldots, x_n$ are distinct points of $[a, b]$, and $y_1, \ldots, y_n$ are any real numbers, there is a unique function g in F such that $g(x_k) = y_k$, $k = 1, \ldots, n$. Define

$$\|f\| = \sup\{|f(x)| : a \leqslant x \leqslant b\},$$

$$f^+(x) = \limsup_{u \to x} f(u), \qquad f^-(x) = \liminf_{u \to x} f(u).$$

If g is in F, define

$$E^+(x; g, f) = f^+(x) - g(x),$$

$$E^-(x; g, f) = f^-(x) - g(x),$$

$$e(g) = \max(\| E^+ \|, \| E^- \|).$$

A function g^* in F is said to be a *best Chebyshev approximation* to f if $e(g^*) \leqslant e(g)$ for all g in F. When f is continuous, this coincides with the classical definition.

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Following Dunham [1], we say that a point $\bar{x}$ in (a, b) is a *straddle point* if for some g in F,

$$E^{+}(\bar{x}) = -E^{-}(\bar{x}) = e(g). \tag{1}$$

Clearly, g is then a best approximation to f, and any other best approximation to f must satisfy (1). If f is continuous, then there are no straddle points unless f belongs to F, in which case all points in (a, b) are straddle points.

The existence of a best Chebyshev approximation to f follows from the usual argument used for proving existence in the continuous case [4]. In case there are *no* straddle points, Remez [6] and Dunham [1] have completely characterized best Chebyshev approximation for bounded functions. Call x^* a "+ point" (− point) relative to $g \in F$ if

$$E^{+}(x^*) = e(g) \qquad (E^{-}(x^*) = -e(g)).$$

Then $g \in F$ is a best approximation to f if there exist, in $[a, b]$, points $x_1 < \cdots < x_{n+1}$ satisfying (i) every x_i is either a + point or a − point, (ii) if x_i is a + point (− point), then x_{i+1} is a − point (+ point), $1 \leqslant i \leqslant n$. Further, if g satisfies this condition, it is the unique best approximation to f.

If, on the other hand, there are straddle points, then the previous characterization and uniqueness theorems no longer hold. For example, let $f(x) = -1$ for $x \leqslant 0, f(x) = 1$ for $x > 0$, and let F consist of all first-degree polynomials on $[-1, 1]$. Then, $x = 0$ is a straddle point, and all polynomials $g(x) = \alpha x, 0 \leqslant \alpha \leqslant 2$, are best approximations to f with $e(g) = 1$. Although $n = 2$, the error curves corresponding to these best approximations alternate only once. However, it is readily seen that if the straddle point is counted as an alternation, then the function $g(x) = 2x$ has error curves with three alternations, and further, every other best approximation still has an error curve with only one alternation

In the following sections, this observation will be generalized, and existence and uniqueness theorems will be proved for the straddle point case.

2. Existence of a Best Approximation with n Alternations

In this section, we assume that f is a real-valued, *regulated* function on $[a, b]$. This means that f is a uniform limit of step-functions. An elementary argument [5] shows that f is regulated if and only if it has left- and right-hand limits at all points of (a, b) and one-sided limits at the end points. In particular, monotone functions and piecewise continuous functions are regulated. Every regulated function has at most countably many discontinuities and is, hence, Riemann integrable.

First, we generalize the concept of alternation of error curves.

DEFINITION 1. A point $\bar{x}$ in (a, b) is a $[-, +]$ point relative to $g \in F$ if

(i) it is a straddle point relative to g, and

(ii) $f^-(\bar{x}) = \lim_{x \to \bar{x}-0} f(x), f^+(\bar{x}) = \lim_{x \to \bar{x}+0} f(x)$.

It is a $[+, -]$ point if (i) holds and if (ii) holds with f^+ and f^- interchanged.

DEFINITION 2. *A function $g \in F$ is said to alternate n times on $[a, b]$ multiplicity counted*, if there exist on $[a, b]$ points $x_1 \leqslant \cdots \leqslant x_{n+1}$ such that

(i) No point occurs more than twice and if $x_i = x_{i+1}$ for some i, $1 \leqslant i \leqslant n$, then x_i is a straddle point;

(ii) Every x_i is a $+$ point or a $-$ point, with the convention that if $x_i = x_{i+1}$ for some i, then x_i is a $+$ $(-)$ point, and x_{i+1} is a $-$ $(+)$ point if and only if x_i is a $[+, -]$ $([-, +])$ point; and

(iii) If x_i is a $+$ $(-)$ point, then x_{i+1} is a $-$ $(+)$ point, $1 \leqslant i \leqslant n$.
The sequence $\{x_i ,..., x_{n+1}\}$ is called a critical point sequence.

It is obvious that if g alternates n times on $[a, b]$, multiplicity counted, then g is a best approximation to f. For, either the critical point sequence has straddle points, in which case $e(g)$ is the smallest Chebyshev error, or there are no straddle points, in which case the characterization result of Dunham applies.

THEOREM 1. *Let f be a regulated function on $[a, b]$. Then there is, in F, a best approximation to f, which alternates n times on $[a, b]$, multiplicity counted.*

Proof. First, assume that f is a step function with one jump discontinuity, i.e.,

$$\begin{aligned} f(x) &= \alpha, \qquad a < x < \bar{x}, \\ &= \beta, \qquad \bar{x} < x < b, \\ \alpha &\neq \beta. \end{aligned} \tag{2}$$

(The values of f at a, b, $\bar{x}$ are immaterial).

Let n_0 be a positive integer such that for

$$n = n_0 , n_0 + 1,..., [\bar{x} - n^{-1}, \bar{x} + n^{-1}] \subseteq [a, b].$$

For these values of n, set

$$\begin{aligned} f_n(x) &= f(x), \qquad x \in I_n , \\ &= \frac{\alpha + \beta}{2} + \frac{n}{2}(\beta - \alpha)(x - \bar{x}), \qquad x \in I_n . \end{aligned} \tag{3}$$

Then each f_n is continuous on $[a, b]$ and, hence, from classical theory [4], f_n has, in F, a unique best approximation g_n. The sequence $\{f_n\}$ is uniformly bounded, for if m is fixed and $n \geqslant m$, then $\| g_n - f_n \| \leqslant \| g_m - f_n \|$ so that, since $\| f_n \| \leqslant | \alpha | + | \beta |$ for all n,

$$\| g_n \| \leqslant \| g_m \| + 2 \| f_n \| \leqslant \| g_m \| + 2(| \alpha | + | \beta |).$$

By a fundamental result of Torhneim [4], there is a subsequence of g_n converging uniformly to a function g in F. For convenience, we denote also the subsequence by g_n. We claim that g satisfies the conditions of the theorem. For let g^* be some fixed best approximation to f and let $e^* = e(g^*)$, $e_n = \| f_n - g_n \|$. Choose n_1 so large that if $n \geqslant n_1$, the oscillation of g^* on I_n is less than $| \beta - \alpha |/2$. Then, for all $n \geqslant n_1$,

$$\| f_n - g^* \| \leqslant e^*. \tag{4}$$

For let y be a point for which $| f_n(y) - g^*(y) | = \| f_n - g^* \|$. If $y \notin I_n$, then $f_n(y) = f(y)$ and (4) follows immediately. If $y \in I_n$, say $y \geqslant \bar{x}$ and $\beta > \alpha$, then either $f_n(y) - g^*(y) = \| f_n - g^* \|$, in which case $f(y) - g^*(y) \geqslant \| f_n - g^* \|$, or $f(y) - g^*(y) = - \| f_n - g^* \|$, in which case

$$\begin{aligned} g^*(\xi) &\geqslant g^*(y) - (\beta - \alpha)/2 \\ &= f(y) + \| f_n - g^* \| - (\beta - \alpha)/2 \\ &\geqslant \alpha + (\beta - \alpha)/2 + \| f_n - g^* \| - (\beta - \alpha)/2 \\ &= \alpha + \| f_n - g^* \| \\ &= f(\xi) + \| f_n - g^* \|, \end{aligned}$$

where ξ is any point in $[\bar{x} - n^{-1}, \bar{x})$. Hence $| g^*(\xi) - f(\xi) | \geqslant \| f_n - g^* \|$, and (4) is proved. The same argument applies if $y \leqslant \bar{x}$ or $\beta < \alpha$. From (4) it follows that for all $n \geqslant n_1$,

$$e_n = \| f_n - g_n \| \leqslant \| f_n - g^* \| \leqslant e^*. \tag{5}$$

On the other hand, if $x \neq \bar{x}$, then $| f_n(x) - g_n(x) | \leqslant e_n$ and, letting n tend to ∞, we obtain $| f(x) - g(x) | \leqslant \lim_n \inf e_n$. Hence

$$e^* = e(g^*) \leqslant e(g) \leqslant \liminf_n e_n. \tag{6}$$

From (5) and (6) we conclude that $e_n \to e^*$ and that $e(g) = e^*$, so that g is a best approximation to f. (In fact, the convergence is monotone: $e_n \leqslant e_{n+1}$.)

It remains to prove that g alternates n times, multiplicity counted. To do this, let $\{x_j^k\}^{n+1}$ be a critical point sequence $j = 1$ for $f_k - g_k$ $(k = n_0, n_0 + 1, \ldots)$. (For fixed k, the x_j^k are distinct, since f_k is continuous.)

Taking subsequences, if necessary, assume that x_1^k is always a $+$ point or always a $-$ point, and that $x_j^k \to x_j$, $j = 1,\dots, n$. Since g_k converges uniformly to g, $g_k(x_j^k) \to g(x_j)$. Hence, if, e.g.,

$$f_k(x_j^k) - g_j^k) = e_k ,$$
$$f_k(x_{j+1}^k) - g_k(x_{j+1}^k) = -e_k ,$$

then

$$\lim_k f_k(x_j^k) - g(x_j) = e^*,$$
$$\lim_k f_k(x_{j+1}^k) - g(x_{j+1}) = -e^*. \tag{7}$$

If x_j and x_{j+1} coalesce, i.e., $x_j = x_{j+1}$, then

$$|\lim_k f_k(x_j^k) - \lim_k f_k(x_{j+1}^k)| = 2e^*. \tag{8}$$

But, then, (8) implies that $|\beta - \alpha| = |f^+(\bar{x}) - f^-(\bar{x})| = 2e^*$, and, hence, $\bar{x}$ is a straddle point. From (7) it is clear that coalescence of three points is impossible. Also, if x_j^k is a $+$ $(-)$ point, $k = 1, 2,\dots$, then, clearly, so is x_j. Hence, g alternates n times, multiplicity counted, and a critical point sequence for it is $\{x_1 ,\dots, x_{n+1}\}$. (As proved above, if there are no straddle points, the x_j are distinct).

If f is a step function with k jump discontinuities, the argument for the existence of g is the same as that for $k = 1$, with an obvious modification in the construction of the f_n.

In the example, $f(x) = 1$, $0 < x \leqslant 1$, $f(x) = -1$, $-1 \leqslant x < 0$, $F =$ the set of second-degree polynomials, the unique best approximation to f_n is $g_n(x) = 2nx/(n + 1)$, $e_n = [2n/(n + 1)] - 1$, and a corresponding critical point sequence is $\{-1, -1/n, 1/n, 1\}$.

Finally, let f be a regulated function on $[a, b]$. There is a sequence $\{s_n\}$ of step functions converging uniformly to f. Hence, there is a sequence $\{g_n\}$ in F such that g_n is a best approximation to s_n, and g_n alternates n times on $[a, b]$, multiplicity counted. Taking subsequences, if necessary, g_n converges uniformly to a function g in F. Since g_n is a best approximation to s_n, g is a best approximation to f. Furthermore, by an argument similar to that employed in the step function case, it follows that g alternates n times, multiplicity counted, completing the proof.

3. Uniqueness of Best Approximations with n Alternations

The example of Section 1 shows that uniqueness of a best Chebyshev approximation is violated when there are straddle points. When such points exist, it is not even true that there is only one best approximation which alternates n times. For, let

$$f(x) = \begin{cases} 3x, & 0 \leqslant x \leqslant 1/3, \\ 6x - 3, & 1/3 \leqslant x < 2/3, \\ 3x - 3, & 2/3 \leqslant x \leqslant 1, \end{cases}$$

and let F consist of all second-degree polynomials. Then $n = 3$, $x = 1/3$ and $x = 2/3$ are straddle points, and any polynomial of the form $p(x) = \alpha(1 - 9x/2 + 9x^2/2)$, $-1 \leqslant \alpha \leqslant 1$, is a best approximation to f, which alternates three times, multiplicity counted.

Clearly, if there are n or more straddle points, then there is a unique best approximation, by the definition of unisolvence. The following is a uniqueness result when $2 \leqslant r \leqslant n - 1$, where r is the number of straddle points. Recall that a double zero is an interior zero at which a sign change does not occur.

LEMMA. *Let f be any bounded function $[a, b]$, g and $\tilde{g}$ members of F. Let $p \geqslant 2$. If $\{x_1, ..., x_p\}$, $x_j < x_{j+1}$, and $\{\tilde{x}_1, ..., \tilde{x}_p\}$, $\tilde{x}_j < \tilde{x}_{j+1}$, are, respectively, on which sequences g and $\tilde{g}$ alternate $p - 1$ times, and if x_1 and $\tilde{x}_1$ are both $+$ $(-)$ points, then $g - \tilde{g}$ has at least zeroes, counting double zeroes twice.*

Proof. Either $x_j = \tilde{x}_j$, $j = 1, ..., p$, and the result is immediate, or there is a smallest j such that $x_j \neq \tilde{x}_j$. Say, $x_j < \tilde{x}_j$.

Now, at a $+$ $(-)$ point for g, $g \leqslant \tilde{g}$ $(g \geqslant \tilde{g})$, and the same statement is true interchanging g and $\tilde{g}$. Then $\{x_1, ..., x_j, \tilde{x}_j, ..., \tilde{x}_p\}$ is a sequence of $p + 1$ distinct points on which $g - \tilde{g}$ consecutively changes sign. By a well-known result [3, p. 61], $g - \tilde{g}$ has at p least zeroes, counting double zeroes twice.

THEOREM 2. *Let f be a regulated function on $[a, b]$, with r straddle points $\bar{x}_1 < \cdots < \bar{x}_r$, $1 \leqslant r \leqslant n - 1$. Suppose there is a g in F such that*

(i) *g alternates n times on $[a, b]$, multiplicity counted;*

(ii) *g does not have more than r alternations on $[\bar{x}_1, \bar{x}_r]$, multiplicity counted.*

Then g is the only best approximation to f with n alternations, multiplicity counted. (Condition (ii) *is always satisfied when $r = 1$.)*

Proof. If g satisfies (i), then g has a critical point sequence which contains all straddle points. So let g satisfy (i) and (ii), with a critical point sequence $x_1 < \cdots < x_k < \bar{x}_1 < \cdots < \bar{x}_r < x_{k+r+1} < \cdots < x_n$. (The proof is the same if $x_1 > \bar{x}_r$ or $x_n < \bar{x}_1$.) Let $\tilde{g}$ satisfy (i).

It suffices to show that $g - \tilde{g}$ has n zeroes, counting double zeroes twice [2]. There are two possibilities:

(a) $\tilde{g}$ satisfies (ii). Let

$$\tilde{x}_1 < \cdots < \tilde{x}_h < \bar{x}_1 < \cdots < \bar{x}_r < \tilde{x}_{h+r+1} < \cdots < \tilde{x}_n$$

be a critical point sequence for $\tilde{g}$. (The proof is the same if $x_1 > \bar{x}_r$ or $x_n < \bar{x}_1$). If $k - h$ is even, the error sign pattern is the same for both g and $\tilde{g}$. By the lemma, $g - \tilde{g}$ has n zeroes, counting double zeroes twice. If $k - h$ is odd, say $k > h$, then $x_1, \ldots, x_k$, $\bar{x}_1, \ldots, \bar{x}_r$, $\tilde{x}_{h+r+1}, \ldots, \tilde{x}_n$ is a sequence on which $g - \tilde{g}$ consecutively changes sign. Then $g - \tilde{g}$ must have at least $n + (k - h) - 1 \geqslant n$ zeroes, counting double zeroes twice. Thus $g = \tilde{g}$.

(b) g has a critical point sequence which contains a point $\bar{y}$ such that $\bar{x}_i < \bar{y} < \bar{x}_{i+1}$ for some i, $1 \leqslant i \leqslant r - 1$, and such that if $\bar{x}_i$ is a $[+, -]$ ($[-, +]$) point, then $\bar{y}$ is a $+$ ($-$) point. Then appending $\bar{y}$ to the critical point sequence for g yields a sequence of $n + 1$ distinct points on which $g - \tilde{g}$ consecutively changes sign. Hence $g = \tilde{g}$.

References

1. C. B. Dunham, Simultaneous Chebychev approximation of functions on an interval, *Proc. Amer. Math. Soc.* **18** (1967), 472–477.
2. J. R. Rice, Tchebycheff approximation by functions unisolvent of variable degree, *Trans. Amer. Math. Soc.* **99** (1961), 298–302.
3. J. R. Rice, "The approximation of functions," Vol. 1, Addison-Wesley, Reading, Mass., 1964.
4. L. Tornheim, On n-parameter families of functions and associated convex functions, *Trans. Amer. Math. Soc.* **69** (1950), 457–467.
5. J. Dieudonné, "Foundations of modern analysis," Academic Press, New York, 1960.
6. E. Ya. Remez, General Computational Methods of Chebyshev Approximation..., Kiev (Russian), 1957: English transl., AEC-tr-4491, U. S. Atomic Energy Commission, 1962.